2006-09-08 Problem Solutions

Problem 1: Tipler 1-30

The formula for the Doppler effect for approach is [Equation (1-38)]

\[
\frac{f}{f_0} = \sqrt{\frac{1 + \beta}{1 - \beta}}.
\]

(1)

In this problem the “proper” wavelength (in the frame of the source) is \(\lambda_0 = 650\,\text{nm}\) (red), and we want to find the speed of approach \(v\) that will shift it to \(\lambda = 590\,\text{nm}\) (green). The ratio of these two wavelengths is \(\lambda/\lambda_0 = 590/650\), so the corresponding frequency ratio is \(f/f_0 = 650/590 = 1.102\) (because \(f = c/\lambda\)). We substitute into Eq. (1) and get

\[
\sqrt{\frac{1 + \beta}{1 - \beta}} = 1.102 \Rightarrow \frac{1 + \beta}{1 - \beta} = (1.102)^2 = 1.214 \Rightarrow 1 + \beta = 1.214(1 - \beta)
\]

\(\Rightarrow 2.214\beta = 0.214 \Rightarrow \beta = 0.0967 \Rightarrow v = 2.90 \times 10^7\,\text{m/s}.
\]

This is the speed at which to approach a red light so that it would appear green. It’s \(6.48 \times 10^7\,\text{miles/hour}\), so you would get a ticket for speeding instead of running a red light.

Problem 2: Tipler 1-32

Since the frequency is increased, the star must be approaching the Earth.

We compare the formula in the problem, telling how much the frequency is increased, \(f = 1.02f_0\), with the equation for the Doppler shift for an approaching object, Eq. (1) above and get

\[
\sqrt{\frac{1 + \beta}{1 - \beta}} = 1.02.
\]

(3)

We have to solve this for \(\beta\). We could solve it by the method that was used in the previous problem: square the equation, clear is of fractions, rearrange it to solve for \(\beta\) and get an answer. But let’s observe that the frequency changes by only 2%, so \(\beta\) must not be very big. Therefore here is an opportunity to use the binomial expansion!

\[
\frac{1 + \beta}{1 - \beta} = (1 + \beta)(1 - \beta)^{-1} = (1 + \beta)[1 + (-1)(-\beta) + \cdots] = (1 + \beta)^2 = 1 + 2\beta + \cdots
\]

(4)

Notice that on the last step we didn’t square out \((1 + \beta)^2\) completely; we dropped the \(\beta^2\) term. That’s because in the binomial expansion step, we already dropped small terms of the order of \(\beta^2\), so to be consistent we have to drop the \(\beta^2\) term here also. Next we take
the square root of Eq. (4), using the binomial expansion again, to get the quantity in Eq. (3).

\[ \sqrt{\frac{1+\beta}{1-\beta}} = [1 + 2\beta + \cdots]^{1/2} = 1 + \frac{1}{2}(2\beta) + \cdots = 1 + \beta. \] (5)

We compare Eqs. (3) and (5) and get

\[ \beta = 0.02 \quad \text{or} \quad v = 0.02c = 6.00 \times 10^6 \text{ m/s}. \] (6)

[Doing the calculation “exactly” without using the binomial expansion gives \( \beta = 0.0198 \).]

**Problem 3: Tipler 1-40**

(a) Alpha Centauri is 4 \( c \cdot y \) away, so the traveler went \( L = \sqrt{1 - \beta^2} (8c \cdot y) \) in 6 \( y \), or

\[ 8c \cdot y \sqrt{1 - v^2/c^2} = v(6y) \]

\[ \sqrt{1 - v^2/c^2} = v(6/8c) = (3/4)(v/c) \]

\[ = 1 - \beta^2 = (3/4)^2 \beta^2 \]

\[ (3/4)^2 \beta^2 + \beta^2 = 1 \]

\[ \beta^2 = 1/(1 + 0.5625) \]

\[ v = 0.8c \]
(b) $\Delta t = \gamma \Delta t_0 = \gamma (6y)$ and $\gamma = 1/\sqrt{1-\beta^2} = 1.667$

$\Delta t = 1.667(6y) = 10y$ or $4y$ older than the other traveler.

(c)
Problem 4: Tipler 1-45

Two events (explosions) in frame S: A \((x_1 = 480 \text{ m}, ct_1 = 0)\); B \((x_2 = 1200 \text{ m}, ct_2 = 1500 \text{ m})\). Here is the spacetime graph with these two events plotted on it.

In frame \(S'\), the events occur at the same point. We draw the straight line between the two events; that’s a line of constant \(x'\), but it isn’t \(x' = 0\) because it doesn’t go through the origin. The slope of this line is

\[
\text{slope}(AB) = \frac{1500 - 0}{1200 - 480} = 2.08.
\]

The line \(x' = 0\), which is the \(ct'\) axis, is parallel to the line \(AB\), so has the same slope. General theory (p. 29) says the slope of the \(ct'\) axis is \(1/\beta\), so

\[
\beta = \frac{1}{2.08} = 0.480.
\]

and then

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 1.140.
\]

To “calibrate” the \(ct'\) axis, we can do the following. Use the Lorentz transformation equation \(t' = \gamma \left( t - \frac{vx}{c^2} \right)\), rewritten as \(ct' = \gamma (ct - \beta x)\), we calculate the time of event A in \(S'\),

\[
ct'_1 = 1.14 \left( 0 - 0.48 \cdot 480 \text{ m} \right) = -263 \text{ m},
\]

and of event B

\[
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\]
\[ ct' = 1.14(1500 \text{ m} - 0.48 \cdot 1200 \text{ m}) = 1053 \text{ m}. \]

Therefore

\[ c\Delta t' = 1053 \text{ m} - (-263 \text{ m}) = 1316 \text{ m} \]

and

\[ \Delta t' = 4.39 \times 10^{-6} \text{ s}. \]

To check, since the events are at the same point in \( S' \), \( \Delta t' \) is a proper time interval and is related to \( \Delta t \) by the time dilation formula.

\[ \Delta t' = \frac{\Delta t}{\gamma} = \frac{5 \times 10^{-6}}{1.14} = 4.39 \times 10^{-6} \text{ s}. \]

It checks.